Problems Curves and Geometry in Space

This material corresponds roughly to sections 12.5, 13.1, 13.2 and 13.4 in the book, as well as the study guide "Curves and geometry"

Problem 1. [this problem will not be evaluated on the exam, only on the written homework ☺] Let $\mathbf{r}(t) = a\cos t\mathbf{i} + b\sin t\mathbf{j}$ for $0 \leq t \leq 2\pi$ parameterize the ellipse $x^2/a^2 + y^2/b^2 = 1$.

a) Compute the curvature $\kappa(t)$ of the curve.

We use the formula for the curvature

$$\kappa(t) = \frac{1}{v^3}|\mathbf{v}(t) \times \mathbf{a}(t)| \tag{1}$$

So we need to compute first the velocity, acceleration and speed

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = -a\sin t\mathbf{i} + b\cos t\mathbf{j} \tag{2}$$

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = -a\cos t\mathbf{i} - b\sin t\mathbf{j} \tag{3}$$

$$v = \|\mathbf{v}(t)\| = \sqrt{(-a\sin t)^2 + (b\cos t)^2} = \sqrt{a^2\sin^2 t + b^2\cos^2 t} \tag{4}$$

Now we compute the cross product of the velocity with the acceleration

$$\begin{aligned}
&\mathbf{v}(t) \times \mathbf{a}(t) \\
=&(-a\sin t, b\cos t, 0) \times (-a\cos t, -b\sin t, 0) \\
=&\det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin t & b\cos t & 0 \\ -a\cos t & -b\sin t & 0 \end{pmatrix} \\
=&(0, 0, ab\sin^2 t + ab\cos^2 t) \\
=&(0, 0, ab(\sin^2 t + \cos^2 t)) \\
=&(0, 0, ab)
\end{aligned}$$

This vector has norm

$$|\mathbf{v}(t) \times \mathbf{a}(t)| = \sqrt{0^2 + 0^2 + (ab)^2} = ab \tag{5}$$

Therefore, the curvature is

$$\kappa(t) = \frac{ab}{\left(\sqrt{a^2 \sin^2 t + b^2 \cos^2 t}\right)^3} = ab\left(a^2 \sin^2 t + b^2 \cos^2 t\right)^{-3/2} \tag{6}$$

b) Suppose that $a > b$. When is the curvature maximal? Try to think geometrically why this must be the case.

We find where $\frac{d\kappa}{dt} = 0$,

$$\begin{aligned}
&\frac{d\kappa}{dt} \\
&= -\frac{3}{2}ab(a^2 \sin^2 t + b^2 \cos^2 t)^{-5/2} \frac{d}{dt}\left(a^2 \sin^2 t + b^2 \cos^2 t\right) \\
&= -\frac{3}{2}ab(a^2 \sin^2 t + b^2 \cos^2 t)^{-5/2}\left(2a^2 \sin t \cos t - 2b^2 \cos t \sin t\right) \\
&= -\frac{3ab\left(a^2 - b^2\right)\sin t \cos t}{\left(a^2 \sin^2 t + b^2 \cos^2 t\right)^{5/2}}
\end{aligned}$$

So the critical points of the curvature function occur when $\frac{d\kappa}{dt} = 0$, or in other words, when $\sin t \cos t = 0$, which can only happen (since $0 \leq t \leq 2\pi$) at the times

$$t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi \tag{7}$$

The values of the curvature at these times are

$$\begin{aligned}
t &= 0 & \kappa(0) &= \frac{ab}{b^3} = \frac{a}{b^2} \\
t &= \frac{\pi}{2} & \kappa(\pi/2) &= \frac{ab}{a^3} = \frac{b}{a^2} \\
t &= \pi & \kappa(\pi) &= \frac{ab}{b^3} = \frac{a}{b^2} \\
t &= \frac{3\pi}{2} & \kappa(3\pi/2) &= \frac{ab}{a^3} = \frac{b}{a^2} \\
t &= 2\pi & \kappa(2\pi) &= \frac{ab}{b^3} = \frac{a}{b^2}
\end{aligned}$$

Now we need to determine which is bigger, $\frac{a}{b^2}$ or $\frac{b}{a^2}$. Notice that if

$$\frac{a}{b^2} <^? \frac{b}{a^2} \tag{8}$$

Then multiplying both sides of the inequality by a^2b^2 we would find that

$$a^3 <^? b^3 \tag{9}$$

and hence

$$a <^? b \tag{10}$$

However, by assumption $a > b$, so in fact the reverse inequality must true. In other words, we have that

$$\boxed{\frac{a}{b^2} > \frac{b}{a^2}} \tag{11}$$

Hence the maximum of the curvature occurs at times

$$t_{max} = 0, \pi, 2\pi \tag{12}$$

Notice that at these times the location of the particle is

$$\begin{cases} \mathbf{r}(0) = (a, 0, 0) \\ \mathbf{r}(\pi) = (-a, 0, 0) \\ \mathbf{r}(2\pi) = (a, 0, 0) \end{cases} \tag{13}$$

In other words, the curvature is the greatest at the points of the ellipse which are further away from the origin.

Problem 2. **[this problem will not be evaluated on the exam, only on the written homework ☺]**

Notice that the graph of a function $y = f(x)$ can be considered as a curve on the xy plane

$$\mathbf{r}(x) = (x, f(x)) \tag{14}$$

Show that the curvature of this curve is always

$$\kappa(x) = \frac{|f''(x)|}{(1 + (f')^2)^{3/2}} \tag{15}$$

We use the same formula as in the previous problem.

$$\kappa(t) = \frac{1}{v^3} |\mathbf{v}(t) \times \mathbf{a}(t)| \tag{16}$$

where we think of using t as the parameter x. In other words, for psychological convenience we write

$$\mathbf{r}(t) = (t, f(t)) \tag{17}$$

So we need to compute first the velocity, acceleration and speed

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = (1, f'(t)) \tag{18}$$

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = (0, f''(t)) \tag{19}$$

$$v = \|\mathbf{v}(t)\| = \sqrt{1 + (f'(t))^2} \tag{20}$$

Therefore

$$
\begin{aligned}
&\mathbf{v}(t) \times \mathbf{a}(t) \\
=&(1, f'(t), 0) \times (0, f''(t), 0) \\
=&\det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & f'(t) & 0 \\ 0 & f''(t) & 0 \end{pmatrix} \\
=&(0, 0, f''(t))
\end{aligned}
$$

This vector has norm

$$|\mathbf{v}(t) \times \mathbf{a}(t)| = \sqrt{0^2 + 0^2 + (f''(t))^2} = |f''(t)| \tag{21}$$

Therefore, the curvature is

$$\kappa(t) = \frac{|f''(t)|}{\left(\sqrt{1 + (f'(t))^2}\right)^3} \tag{22}$$

Replacing t with x, we obtain the formula the problem asked us to find.

Problem 3. [this problem will not be evaluated on the exam, only on the written homework ☺]

Compute the curvature κ for the twisted cubic

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k} \tag{23}$$

We use the same formula as in the previous problem.

$$\kappa(t) = \frac{1}{v^3}|\mathbf{v}(t) \times \mathbf{a}(t)| \tag{24}$$

So we need to compute first the velocity, acceleration and speed

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = (1, 2t, 3t^2) \tag{25}$$

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = (0, 2, 6t) \tag{26}$$

$$v = \|\mathbf{v}(t)\| = \sqrt{1 + (2t)^2 + (3t^2)^2} = \sqrt{1 + 4t^2 + 9t^4} \tag{27}$$

Therefore

$$
\begin{aligned}
&\mathbf{v}(t) \times \mathbf{a}(t) \\
=&(1, 2t, 3t^2) \times (0, 2, 6t) \\
=&\det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{pmatrix} \\
=&(12t^2 - 6t^2, -6t, 2) \\
=&(6t^2, -6t, 2)
\end{aligned}
$$

This vector has norm

$$|\mathbf{v}(t) \times \mathbf{a}(t)| = \sqrt{36t^4 + 36t^2 + 4} \tag{28}$$

Therefore, the curvature is

$$\kappa(t) = \frac{\sqrt{36t^4 + 36t^2 + 4}}{\left(\sqrt{1 + 4t^2 + 9t^4}\right)^3} \tag{29}$$

Replacing t with x, we obtain the formula the problem asked us to find.

Problem 4. Let $\mathbf{r}(t)$ represent the trajectory of a curve. Show that at a local maximum or minimum of $f(t) = \|\mathbf{r}(t)\|$, we have that $\mathbf{r}(t)$ is perpendicular to $\frac{d}{dt}\mathbf{r}(t)$. [Hint: check part a) of Problem 1].

At a local maximum or minimum we should have $f'(t) = 0$, in other words

$$\frac{d}{dt}\|\mathbf{r}(t)\| = 0 \tag{30}$$

From part a) of Problem 1 we know that

$$\frac{d}{dt}\|\mathbf{r}\| = \frac{1}{\|\mathbf{r}\|}\mathbf{r} \cdot \frac{d\mathbf{r}}{dt} \tag{31}$$

so we can conclude that

$$\mathbf{r} \cdot \frac{d\mathbf{r}}{dt} = 0 \tag{32}$$

at any time t which corresponds to a critical point of $f(t)$.